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Abstract
It is shown that the radial part of the hydrogen Hamiltonian factorizes as the product of two not mutually adjoint first-order differential operators plus a complex constant ϵ. The 1-SUSY approach is used to construct non-Hermitian operators with hydrogen spectra. Other non-Hermitian Hamiltonians are shown to admit an extra 'complex energy' at ϵ. New self-adjoint hydrogen-like Hamiltonians are also derived by using a 2-SUSY transformation with complex conjugate pairs $\epsilon, \bar{\epsilon}$.

PACS numbers: 03.65.Ge, 03.65.Fd, 03.65.Ca

1. Introduction

The supersymmetric quantum mechanics (SUSY QM) has grown on the factorization and intertwining methods [1] applied to transform the physical Hamiltonians. It yields new exactly solvable potentials which are either strictly isospectral to the initial one because of broken SUSY, or almost isospectral due to unbroken SUSY (see the recent reviews [2]). The higher order SUSY QM amended the conviction that the excited states cannot be used to generate non-singular SUSY partners [3–6]. Some applications deal with singular [7], soliton-type [8], periodic [9] and other potentials [10]. Of special interest is the confluent algorithm [8, 11] for which the second-order procedure is applied to add a single level at an arbitrary point of the energy axis. However, almost all the works on the subject make use of transformation functions with the real factorization constants ϵ and with the factor-operators being always mutually adjoint.

The case of complex ϵ has not been studied to the desired extent. Exceptional cases are [12–14], where the 2-SUSY treatment with $\epsilon \in \mathbb{C}$ is formulated to obtain either Hermitian or non-Hermitian SUSY partners of a given initial Hamiltonian. Indeed, one of our purposes is to show that the Hamiltonians H can be factorized as the product of two not mutually adjoint first-order operators A, B, plus a complex constant even for self-adjoint H. The method is

not limited to the Hamiltonians possessing the lower spectral bound; it can also yield non-Hermitian Hamiltonians with complex potentials. Although the typical Hamiltonians of QM are Hermitian, non-Hermitian ones appear in molecular physics and quantum chemistry [15], superconductivity [16], quantum field theory [17] and other domains [18]. The fact that they admit real eigenvalues for which the associated eigenfunctions are square-integrable [19] has been the basis of recent studies on PT-symmetry [20, 21], pseudo-Hermiticity [22, 23] and diverse physical models [24].

In this paper, we illustrate these facts by constructing the Hermitian and non-Hermitian SUSY partners of the radial part (H_ℓ) of the hydrogen Hamiltonian. Even though our non-Hermitian operators are not PT-invariant, we shall see that a class of them has real eigenvalues identical to the hydrogen energies. The reality of the spectrum in this case is due to the breaking of supersymmetry.

It will be shown that there is another class of non-Hermitian operators having an extra square-integrable eigenfunction associated with $\epsilon \in \mathbb{C}$. In this case, the 'complex energy' ϵ arises from the unbroken supersymmetry and, up to now, does not have a well-established physical meaning (but see [25]). Unlike the phase-equivalent complex potentials [14], the new 'bound state' associated with ϵ is nodeless. Moreover, in counterdistinction with the formalism of PT-symmetry and pseudo-Hermiticity, where complex energies appear in conjugate pairs, it turns out that $\bar{\epsilon}$ does not belong to the spectrum of the SUSY partner of H_ℓ generated through ϵ.

In general, we shall see that the SUSY transformation is adequate to analytically determine normalizable eigenfunctions of non-Hermitian Hamiltonians, including those with complex energies. In this sense, our 'complex SUSY transformation' seems an analytical complement of the numerical techniques previously reported (see [14] and references therein). The eigenfunction connected with ϵ is then removed by iterating the procedure in order to construct Hermitian 2-SUSY partners of H_ℓ.

The paper is organized as follows: section 2 introduces the atypical factorizations $H_\ell = AB + \epsilon$, where $\epsilon \in \mathbb{C}$ and the first-order differential operators A and B are not mutually adjoint. Sections 3 and 4 are devoted to the construction of non-Hermitian 1-SUSY and Hermitian 2-SUSY partners, $\mathsf{H}(\zeta)$ and $\widetilde{H}$ respectively, of H_ℓ. Final remarks and discussion are given in section 5.

2. The complex-type factorization method

Let us consider a single electron in the field produced by a nucleus with Z protons. We shall use $\mathcal{E} = Z/2r_B$ and $r_B = \hbar^2/Ze^2m$ for the units of energy and coordinates, respectively. The corresponding time-independent Schrödinger equation reduces to $H_\ell \psi_{n,\ell}(r) = E_n \psi_{n,\ell}(r)$, with solutions

$$\psi_{n,\ell}(r) = C_{n,\ell} r^{\ell+1} e^{-r/n} {}_1F_1(\ell + 1 - n, 2\ell + 2; 2r/n) \qquad E_n = -1/n^2 \quad (1)$$

where $\mathbb{N} \ni n = \ell + s + 1$; $\ell = 0, 1, 2, \ldots, n - 1$; $s \in \mathbb{Z}^+$; $C_{n,\ell}$ is the normalization constant, ${}_1F_1(a, c, z)$ is Kummer's function and $L^2(\mathbb{R}^+, 4\pi) \ni \psi_{n,\ell}(r) \equiv rR_{n,\ell}(r)$, with an inner product defined by $\langle \psi, \phi \rangle = 4\pi \int_0^{+\infty} \bar{\psi}(r)\phi(r)\,dr < \infty$ and boundary conditions at $r = 0$: $\psi(0) = 0$, $\psi'(r) = R(0)$. The effective potential $V_\ell(r)$ has the domain $\mathcal{D}_V = [0, \infty)$ and

$$H_\ell \equiv -\frac{d^2}{dr^2} + V_\ell(r) = -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} - \frac{2}{r}. \qquad (2)$$

The Hamiltonian (2) is factorized as follows:

$$H_\ell = AB + \epsilon \qquad (3)$$

where the factorization constant is a complex number $\mathbb{C} \ni \epsilon := \epsilon_1 + i\epsilon_2; \epsilon_1, \epsilon_2 \neq 0 \in \mathbb{R}$ and the first-order operators A, B, are not mutually adjoint (compare [6, 26]):

$$A := -\frac{d}{dr} + \beta(r) \qquad B := \frac{d}{dr} + \beta(r) \tag{4}$$

with β a complex-valued function fulfilling

$$-\beta'(r) + \beta^2(r) + \epsilon = V_\ell(r). \tag{5}$$

The Riccati equation (5) is solved by means of the logarithmic transformation $\beta(r) = -\frac{d}{dr} \ln u(r)$ for which u is the most general eigenfunction of H_ℓ (not necessarily normalizable) belonging to $\epsilon \equiv -k^2; \mathbb{C} \ni k = k_1 + ik_2; k_1, k_2 \in \mathbb{R}$:

$$
\begin{aligned}
u(r) &= r^{\ell+1} e^{-kr} f(r) \\
f(r) &:= \alpha_1 F_1(\ell + 1 - 1/k, 2\ell + 2, 2kr) + \zeta U(\ell + 1 - 1/k, 2\ell + 2, 2kr)
\end{aligned}
\tag{6}
$$

where α and ζ are complex constants and $U(a, c, z)$ is the logarithmic hypergeometric function. The global behaviour of these u-functions is analysed in the appendix.

Hence, for the β-function we have

$$\beta(r) = -\frac{\ell+1}{r} + k + \Omega(r) \qquad \Omega(r) := -\frac{d}{dr} \ln f(r). \tag{7}$$

A convenient expression for $\beta_1(r), \beta_2(r)$, the real and imaginary parts of $\beta(r)$ respectively, can be found in the appendix.

3. New complex hydrogen-like potentials

Let us consider the value ϵ fixed. For convenience we shall make explicit the dependence of Ω on ζ. Now, let us reverse the order of the factors in (3):

$$BA + \epsilon = -\frac{d^2}{dr^2} + V_{\ell+1}(r) + 2\Omega'(r; \zeta) \equiv -\frac{d^2}{dr^2} + V(r; \zeta) := H(\zeta) \tag{8}$$

where $H(\zeta)$ is a non-Hermitian second-order differential operator and we have used (5) and (7). The next step is to solve the related eigenvalue equation:

$$H(\zeta)\Psi = \lambda\Psi \qquad \lambda = \lambda_1 + i\lambda_2 \qquad \lambda_1, \lambda_2 \in \mathbb{R}. \tag{9}$$

The dependence of Ψ and λ on ζ will be dropped for simplicity. Note that equations (3) and (8) imply an intertwining between the Hamiltonian H_ℓ and the non-Hermitian operator $H(\zeta)$:

$$H(\zeta)B = BH_\ell \qquad H_\ell A = AH(\zeta). \tag{10}$$

Thereby, one sees that $\Psi \propto B\varphi$ is a solution of (9) if φ satisfies $H_\ell\varphi = \lambda\varphi$, while A reverses the action of B. Now, the general form of φ is obtained by taking $\epsilon = -k^2$ for $\lambda = -\kappa^2$, and $u(r)$ for $\varphi(r)$ in (6). Hence, we have

$$\varphi(r) = r^{\ell+1} e^{-\kappa r}\{C_1 F_1(\ell + 1 - 1/\kappa, 2\ell + 2, 2\kappa r) + DU(\ell + 1 - 1/\kappa, 2\ell + 2, 2\kappa r)\} \tag{11}$$

with C and D arbitrary complex constants. Therefore,

$$\Psi \propto B\varphi = \frac{W(u, \varphi)}{u}. \tag{12}$$

We are looking now for the constraints on $\alpha, \zeta, \lambda, C$ and D leading to square-integrable Ψ. First, consider $\lambda \neq \epsilon$ (i.e., $\kappa \neq k$); the behaviour of Ψ near the origin is

$$
\Psi(r \sim 0) \propto
\begin{cases}
-D\frac{(2\ell+1)}{(2\kappa)^{2\ell+1}} \frac{\Gamma(2\ell+1)}{\Gamma(\ell+1-1/\kappa)} \frac{1}{r^{\ell+1}} & \zeta = 0 \quad \alpha \neq 0 \\
D\frac{(k-\kappa)}{(2\kappa)^{2\ell+1}} \frac{\Gamma(2\ell+1)}{\Gamma(\ell+1-1/\kappa)} \frac{1}{r^\ell} & \zeta \neq 0 \quad \text{arbitrary } \alpha.
\end{cases}
\tag{13}
$$

Thus, Ψ becomes divergent at $r = 0$ except if either (1) $\lambda = -\kappa^2$ is real and $\kappa^{-1} = \ell + s + 1$, $s \in \mathbb{Z}^+$, or (2) λ is complex but $D = 0$. Let us pay some attention to these conditions.

Case 1 (real λ). Let us fix $\mathbb{R} \ni \kappa^{-1} = \ell + s + 1 = n, n \in \mathbb{N}$. In this case, in (11) $_1F_1(a, c; z)$ and $U(a, c; z)$ are essentially the same function. Therefore, one can take $D = 0$ and $C = C_{n,\ell}$ (see equation (1)), so $\varphi(r) = \psi_{n,\ell}(r)$ and the functions (12) behave asymptotically as $\Psi(r) \propto \psi_{n,\ell}(r)_{r \to \infty}$. On the other hand, a straightforward calculation shows that these functions obey the following boundary conditions at $r = 0$: $\Psi(0, \zeta) = R(0)$; $\Psi'(0, \zeta = 0) = -\delta_{\ell 0}$; $\Psi'(0, \zeta \neq 0) = -\ell\left(\frac{\delta_{\ell 0}}{r} + \delta_{\ell 1}\right)$, with $\delta_{\ell n}$ the Kronecker delta. Thus

$$\Psi(r; \zeta) \propto \left[k - \frac{1}{n} + \frac{\mathrm{d}}{\mathrm{d}r} \ln\left(\frac{_1F_1(\ell + 1 - n, 2\ell + 2; 2r/n)}{f(r)}\right)\right] \psi_{n,\ell}(r) \qquad (14)$$

are square-integrable eigenfunctions of $\mathsf{H}(\zeta)$ with the real eigenvalues $\lambda = -1/n^2 = E_n$.

Case 2 (complex λ). For $\kappa \in \mathbb{C}$ and $D = 0$ the function (12) behaves asymptotically as follows:

$$\Psi(r) \sim \begin{cases} C\frac{(\kappa \mp k)}{(2\kappa)^{\ell+1+1/\kappa}}\frac{\Gamma(2\ell+2)}{\Gamma(\ell+1-1/\kappa)}\frac{\mathrm{e}^{\kappa r}}{r^{1/\kappa}} & \kappa_1 > 0 \quad \begin{cases} k_1 > 0 \\ k_1 < 0 \end{cases} \\ \mp C\frac{(k \pm \kappa)}{(-2\kappa)^{\ell+1-1/\kappa}}\frac{\Gamma(2\ell+2)}{\Gamma(\ell+1+1/\kappa)}\frac{\mathrm{e}^{-\kappa r}}{r^{-1/\kappa}} & \kappa_1 < 0 \quad \begin{cases} k_1 > 0 \\ k_1 < 0 \end{cases} \end{cases} \qquad (15)$$

which always diverges for $r \to \infty$. Hence, there is no function $\Psi \in L^2(\mathbb{R}^+, 4\pi)$ solving (9) for a complex $\lambda \neq \epsilon$ (i.e., $\kappa \neq k$) and $D = 0$.

Formula (14) therefore gives all square-integrable solutions $\{\Psi(r; \zeta)\}$ of (9) for $\lambda \neq \epsilon$. Concerning the case $\lambda = \epsilon$, we see from equation (8) that any $\Psi_\epsilon(r)$ in the one-dimensional kernel of A is an eigenfunction of $\mathsf{H}(\zeta)$ belonging to ϵ. After a simple calculation one gets

$$\Psi_\epsilon(r) \propto \frac{1}{u(r)}. \qquad (16)$$

This function can be in $L^2(\mathbb{R}^+, 4\pi)$ for appropriate values of α and ζ. In such a case, Ψ_ϵ is an extra square-integrable eigenfunction of $\mathsf{H}(\zeta)$ associated with $\epsilon \in \mathbb{C}$. Finally, a straightforward calculation shows that, although $\{\Psi(r; \zeta), \Psi_\epsilon(r)\}$ are elements of $L^2(\mathbb{R}^+, 4\pi)$, they do not form an orthogonal set. A discussion on these kinds of properties of the inner product is given in [24]. The next subsections analyse these conditions and classify the resulting potentials according to their spectra.

3.1. The real spectrum

From figure 1(*a*), one sees that the behaviour of the complex function $\mathsf{V}(r; \zeta)$, for $\alpha \neq 0$ and $\zeta = 0$, is given by

$$\mathsf{V}(r; \zeta = 0) \sim \begin{cases} V_{\ell+1}(r) & \text{for} \quad r \sim 0 \\ 0 & r \to \infty. \end{cases}$$

Consistently with case (a) of the appendix, the function $\Psi_\epsilon(r)$ diverges at $r = 0$. Hence, $\Psi_\epsilon(r)$ is not in $L^2(\mathbb{R}^+, 4\pi)$ and the discrete spectrum $\sigma_d(\mathsf{H}_{\ell+1})$ of $\mathsf{H}(\zeta = 0)$ is exactly the same as that of the hydrogen atom $\sigma_d(\mathsf{H}_{\ell+1}) = \sigma_d(H_\ell)$. Figure 1(*b*) depicts the behaviour of $\psi_{n,\ell}$ and the related function (14) for one of the excited states.

Another class of complex potentials $\mathsf{V}(r; \zeta)$ sharing the same spectrum as H_ℓ is obtained by considering case (b) of the appendix.
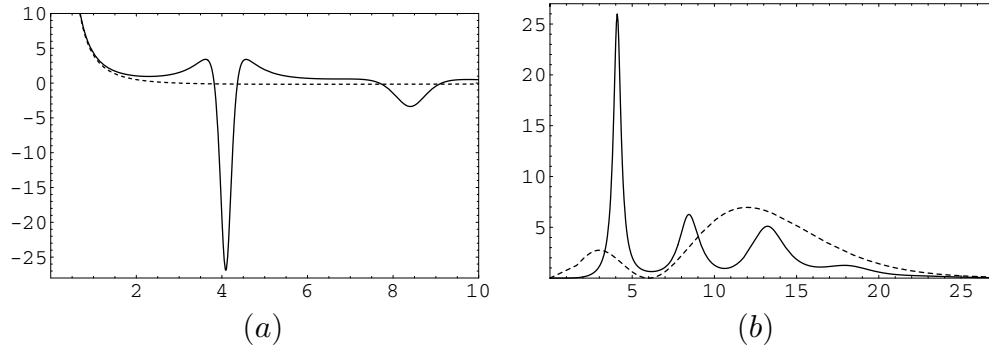
Figure 1. (a) The potential $V_{\ell+1}(r)$ (dashed curve) and the real part of $V(r; \zeta)$ for $\ell = 1$, $\epsilon = -(0.1 + i\,0.5)^2$, $\alpha = 1$ and $\zeta = 0$. (b) The corresponding unnormalized $|\psi_{3,1}(r)|^2$ (dashed curve) and its SUSY partner $|\Psi(r; \zeta = 0)|^2$ for $E_3 = -1/9$ and the same values of the parameters.
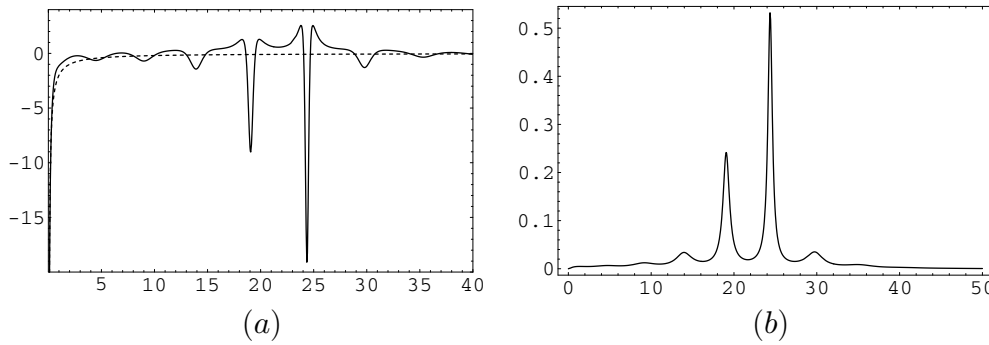

Figure 2. (a) The potential $V_{\ell-1}(r)$ (dashed curve) and the real part of $V(r; \zeta)$, for $\ell = 1$, $\epsilon = -(0.1 + i\,0.5)^2$, $\alpha = 1$ and $\zeta = 0.5 + i\,0.5$. (b) The corresponding unnormalized $|\Psi_{\epsilon}(r)|^2$ for the same values of the parameters.

3.2. Complex potentials admitting complex 'energies'

Let us consider $\zeta \neq 0$. Then the complex function $V(r; \zeta)$ behaves as shown in figure 2(a), i.e.

$$V(r; \zeta \neq 0) \sim \begin{cases} V_{\ell-1}(r) & \text{for} \quad r \sim 0 \quad \text{arbitrary } \alpha \\ 0 & r \to \infty. \end{cases}$$

Case (d) of the appendix shows that, for $(\alpha, \zeta) \in \mathbb{C}_0$ and $\theta_\ell(k) \neq 0$, the function $\Psi_\epsilon(r)$ behaves as

$$\Psi_\epsilon(r) \sim \begin{cases} \frac{(2k)^{2\ell+1}}{\zeta} \frac{\Gamma(\ell+1-1/k)}{\Gamma(2\ell+1)} r^\ell & r \sim 0 \\ \frac{(2k)^{\ell+1+1/k}}{\alpha} \frac{\Gamma(\ell+1-1/k)}{\Gamma(2\ell+2)} r^{1/k}\, e^{-kr} & r \to \infty \quad k_1 > 0 \\ \frac{(2k)^{\ell+1-1/k}}{\theta_{\ell(k)}} r^{-1/k}\, e^{kr} & r \to \infty \quad k_1 < 0 \end{cases} \tag{17}$$

which clearly belongs to $L^2(\mathbb{R}^+, 4\pi)$. Figure 2(b) depicts the global behaviour of $|\Psi_\epsilon|^2$. Therefore, the discrete spectrum $\sigma_d(\mathsf{H}_{\ell-1})$ of $\mathsf{H}(\zeta \neq 0)$ is given by $\sigma_d(\mathsf{H}_{\ell-1}) = \sigma_d(H_\ell) \cup \{\epsilon\}$, provided that $(\alpha, \zeta) \in \mathbb{C}_0, \theta_\ell(k) \neq 0$.

Finally, case (c) of the appendix, gives another solution in the same class.

4. The new real hydrogen-like potentials

We shall now extend the analysis of the previous section by intertwining H_ℓ with a new (to be determined) Hamiltonian $\widetilde{H}$ as follows:

$$\widetilde{H}\widetilde{A} = \widetilde{A}H_\ell \tag{18}$$

where the differential operator $\widetilde{A}$ is of the second order

$$\widetilde{A} := \frac{d^2}{dr^2} + \eta(r)\frac{d}{dr} + \gamma(r) \tag{19}$$

and $\widetilde{H}$ reads

$$\widetilde{H} := -\frac{d^2}{dr^2} + \widetilde{V}(r). \tag{20}$$

The operators (19) and (20) depend implicitly on the label ℓ. A straightforward calculation allows us to express the functions η and γ of (19) in terms of the auto-Bäcklund transformation of the solutions of (5) for ϵ_a and ϵ_b [6]:

$$\eta(r) = -\left(\frac{\epsilon_a - \epsilon_b}{\beta_a(r) - \beta_b(r)}\right) \qquad \epsilon_a \neq \epsilon_b \tag{21}$$

$$\gamma(r) = \beta_b'(r) - \beta_b^2(r) + \eta(r)\beta_b(r) \qquad \epsilon_a \neq \epsilon_b. \tag{22}$$

Thus, the second-order intertwining operator $\widetilde{A}$ in (19) is expressed by two different solutions of the first-order case. Moreover, it factorizes as $\widetilde{A} = a_2 a_1$, where

$$a_1 \equiv \frac{d}{dr} + \beta_a = B \qquad a_2 \equiv \frac{d}{dr} + \eta - \beta_a = -A + \eta. \tag{23}$$

Thereby, it is easy to rewrite $\widetilde{A}$ as

$$\widetilde{A} = (-A + \eta)B = -H_\ell + \epsilon + \eta B \tag{24}$$

and potential (20) is obtained through

$$\widetilde{V}(r) = V_\ell(r) + 2\eta'(r). \tag{25}$$

In order to get a real $\eta(r)$ in (21), (25), we consider the solution $\beta_a(r)$ of (5) for $\epsilon_a \in \mathbb{C}$ as given and, by taking $\epsilon_b = \bar{\epsilon}_a$ and $\beta_b(r) = \bar{\beta}_a(r)$ in (21), one finds

$$\eta(r) = -\frac{\text{Im}(\epsilon_a)}{\text{Im}(\beta_a)} \equiv -\frac{\epsilon_2}{\beta_2(r)} = -\frac{d}{dr}\ln\omega(r) \tag{26}$$

where ω is defined in (A.2) of the appendix and the labels a and b have been dropped from ϵ_2 and β_2. Henceforth, potential (25) is real:

$$\widetilde{V}(r) = V_\ell(r) - 2\frac{d^2}{dr^2}\ln\omega(r). \tag{27}$$

We are looking for potentials $\widetilde{V}(r)$ defined in the same initial domain $\mathcal{D}_V = [0, \infty)$ (the situation when the initial domain is changed requires a different treatment, see, e.g., Márquez *et al* in [7]). According to the proposition in the appendix, ω has at most one isolated zero in $\mathcal{D}_V$. By choosing a proper u, the function ω can be so constructed that its isolated zero coincides with one of the edges of $\mathcal{D}_V$ (compare [13]):

$$\lim_{r\to 0} u(r) = 0 \quad \Rightarrow \quad \lim_{r\to 0}\omega(r) = 0 \tag{28}$$

$$\lim_{r\to\infty} u(r) = 0 \quad \Rightarrow \quad \lim_{r\to\infty}\omega(r) = 0. \tag{29}$$
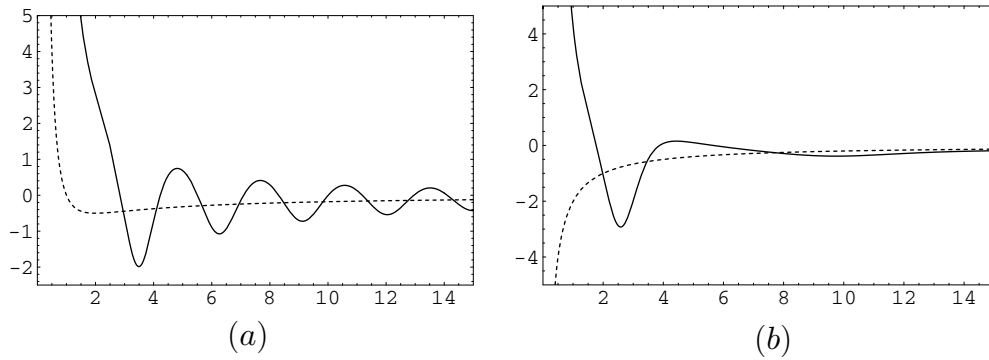
Figure 3. The initial potential $V_\ell(r)$ (dashed curve) and its 2-SUSY partner $\widetilde{V}(r)$ for $\alpha = 1$, $\zeta = 0$ and (a) $\ell = 1$, $\epsilon = -(0.01 + \mathrm{i})^2$, (b) $\ell = 0$, $\epsilon = -(0.5 + \mathrm{i}\,0.1)^2$.

Let us examine the consequences. First, condition (28) is satisfied if the u in (6) is chosen with $\alpha \neq 0$ and $\zeta = 0$, so that the potential (27) behaves as shown in figure 3, i.e.

$$\widetilde{V}(r; \alpha \neq 0; \zeta = 0) \sim \begin{cases} V_{\ell+2}(r) & \text{for} \quad r \sim 0 \quad \text{arbitrary } k_1 \\ 0 & r \to \infty \quad \text{arbitrary } k_1. \end{cases} \tag{30}$$

On the other hand, condition (29) can be achieved for $\alpha = 0$, $\zeta \neq 0$ and $k_1 > 0$. In this case equation (27) leads to

$$\widetilde{V}(r; \alpha = 0; \zeta \neq 0) \sim \begin{cases} V_\ell(r) & \text{for} \quad r \sim 0, \\ 0 & r \to \infty. \end{cases} \tag{31}$$

The real-valued potentials (30) and (31) resemble those of the hydrogen and they could, in principle, represent physical systems as the Hamiltonian (20) is self-adjoint. The next step is to analyse the new eigenvalue equation

$$\widetilde{H}\widetilde{\psi} = \widetilde{E}\widetilde{\psi} \tag{32}$$

whose solutions, by similar arguments as for the first-order case, are now obtained from the linear second-order transformation (see equation (18)):

$$\widetilde{\psi}(r) \propto \widetilde{A}\psi_{n,\ell}(r) = (-E_n + \epsilon)\psi_{n,\ell}(r) + \eta(r, \zeta)\Psi(r, \zeta) \tag{33}$$

where we have used (24) and $\Psi(r, \zeta)$ is given by (14). The corresponding boundary conditions at $r = 0$ can be obtained from those of $\psi_{n,\ell}(r)$ and $\Psi(r, \zeta)$.

It is clear that the first term on the rhs of (33) is in $L^2(\mathbb{R}^+, 4\pi)$. The behaviour of the related second term is found by observing that

(1) If $\alpha \neq 0$ and $\zeta = 0$ then η diverges as r^{-1} at the origin while it is constant at $r = \infty$. Therefore, the product $\eta(r, 0)\Psi(r, 0)$ is zero at the edges of $\mathcal{D}_V$ and remains finite in all $\mathcal{D}_V$.

(2) If $\alpha = 0$ and $\zeta \neq 0$ then η is a constant at both edges of $\mathcal{D}_V$. Hence, $\eta(r, \zeta \neq 0)\Psi(r, \zeta \neq 0)$ is again well behaved in all $\mathcal{D}_V$.

Thus, the eigenfunctions $\widetilde{\psi}$ of $\widetilde{H}$ given in (33) are in $L^2(\mathbb{R}^+, 4\pi)$ and $\widetilde{H}$ is an exactly solvable Hamiltonian with the same spectrum as the hydrogen atom.

5. Summary and discussion

In this paper, we have used a new type of factorization method to analyse a set of non-Hermitian SUSY partners of the radial part of the hydrogen Hamiltonian. In order to generate the corresponding potentials we have used Darboux transformations with complex factorization constants. In contrast to PT-symmetry [19–21] and pseudo-supersymmetry [22], the breaking of supersymmetry leads to purely real spectra. However, while the pseudo-Hermiticity and PT-symmetry breaking involve pairs of conjugate complex eigenvalues, the unbroken supersymmetry (for which the non-Hermitian 1-SUSY partners are not strictly isospectral) involves just a single 'complex energy'. To be more precise, in order to add two extra eigenvalues (real or complex) to a given spectrum one applies either twice the 1-SUSY procedure or a single 2-SUSY transformation (both can be made equivalent for the case we are dealing with). Now, if the two new energies form a complex conjugate pair and if the functions $\eta(r)$ and $\gamma(r)$ of the non-singular intertwining operator (19) are real, then the final Hamiltonian becomes self-adjoint and does not admit any complex eigenvalue, just as we have shown in section 4.

The problem of finding normalizable solutions belonging to complex eigenvalues for non-Hermitian Schrödinger equations has been solved previously by numerical techniques in [14] and analysed inside a Lie-algebraic framework in [27]. In this paper, we performed an analytical study and we expect that our results complement the numerical ones. Similarly as in other non-Hermitian cases discussed in the literature, the interpretation of the 'complex energies' is an open problem, though we note possible applications to the absorptive (dissipative) systems [25].

In general, we have shown that the extension of the SUSY treatment to include complex factorization constants leads to results which are out of the scope of the PT-symmetry and pseudo-Hermiticity. Indeed, the reality of the spectrum of the Hamiltonians $H(\zeta)$ in section 3 depends on the parameter ζ, as has been established in subsections 3.1 and 3.2. As the non-Hermitian Hamiltonians $H(\zeta \neq 0)$ in subsection 3.2 do not satisfy the theorems by Mostafazadeh [22], they are not pseudo-Hermitian. On the other hand, it is not yet clear whether the Hamiltonians $H(\zeta = 0)$ of subsection 3.1 could be pseudo-Hermitian or not (though a primary impression can be depicted by noting that the non-Hermiticity of $H(\zeta)$ depends not on ζ but on the non-trivial imaginary part of the factorization constant ϵ). Work in this direction is in progress and will be published elsewhere.

Appendix

The global behaviour of the eigenfunctions $u(r)$ of equation (6) can be described in terms of α, ζ and the sign of k_1:

(a) If $\alpha \neq 0$ and $\zeta = 0$ then $u(r)$ is zero at the origin $r = 0$ while it diverges at $r = \infty$.
(b) If $\alpha = 0$, $\zeta \neq 0$ and $k_1 > 0$, then $u(r)$ diverges at the origin and tends to zero for $r \to \infty$.
(c) If $\alpha = 0$, $\zeta \neq 0$ and $k_1 < 0$, then $u(r)$ diverges at $r = 0$ and $r = \infty$.
(d) Let $\mathbb{C}_0 \subset \mathbb{C} \times \mathbb{C}$ be the subset of complex pairs $\alpha \neq 0$, $\zeta \neq 0$, such that

$$\zeta \neq -\alpha \frac{{}_1F_1(\ell + 1 - 1/k, 2\ell + 2, 2kr_0)}{U(\ell + 1 - 1/k, 2\ell + 2, 2kr_0)} \qquad \forall r_0 \in (0, \infty).$$

If

$$\theta_\ell(k) := \zeta + \alpha \frac{\Gamma(2\ell + 2)}{\Gamma(\ell + 1 + 1/k)} \, \mathrm{e}^{\pm \mathrm{i}(\ell + 1 - 1/k)\pi} \tag{A.1}$$

is different from zero for $(\alpha, \zeta) \in \mathbb{C}_0$, then $u(r)$ is free of zeros in all $\mathcal{D}_V$ and diverges at $r = 0$ and $r = \infty$.

(e) If $(\alpha, \zeta) \in \mathbb{C}_0, k_1 < 0$ and $\theta_\ell(k) = 0$, then $u(r)$ diverges at the origin while $\lim_{r \to \infty} u(r) = 0$ (coinciding indeed with (b)).

The presence of zeros in these functions has been studied by means of the following:

Proposition. *Let $u(r) \in C^1(D_v)$ be a solution of the Schrödinger equation $u''(r) = [v(r) - \epsilon]u(r)$, where $v(r)$ is a real-valued potential with domain $\mathcal{D}_v$ and $\epsilon \in \mathbb{C}$. Assume that $\mathcal{D}_v$ is a simply connected region of $\mathbb{R}$. If $\mathrm{Im}(\epsilon) \neq 0$, then the complex-valued function $u(r)$ admits at most one isolated zero in $\mathcal{D}_v$.*

Proof. Let

$$\omega(r) := \frac{W(u, \bar{u})}{2\mathrm{i} \, \mathrm{Im}(\epsilon)} \tag{A.2}$$

where the bar denotes complex conjugation and $W(\cdot, \cdot)$ corresponds to the Wronskian of the involved functions. Clearly ω is continuous in $\mathcal{D}_v$ and $\omega'(r) = |u(r)|^2 \geqslant 0 \, \forall r \in \mathcal{D}_v$, so $\omega(r)$ is always non-decreasing and can have either only one isolated zero or an entire interval of zero points in $\mathcal{D}_v$. As every zero of $u(r)$ is, necessarily, a zero of $\omega(r)$ then $u(r)$ admits at most one isolated zero there. $\qquad \square$

The real function $\omega(r)$ in (A.2) plays a relevant role in the 2-SUSY approach of section 4. A convenient expression for $\beta(r)$ in terms of ω is given by

$$\beta(r) = \beta_1(r) + \mathrm{i}\beta_2(r) \equiv -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\ln\omega'(r) + \mathrm{i}\epsilon_2\left(\frac{\mathrm{d}}{\mathrm{d}r}\ln\omega(r)\right)^{-1}. \tag{A.3}$$

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